Knots and Matrix Models

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Abstract

We consider a matrix model with d matrices $N \times N$ and show that in the limit $N \to \infty$ and $d \to 0$ the model describes the knot diagrams. The same limit in matrix string theory is also discussed. We speculate that a prototypical M(atrix) without matrix theory exists in void.

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1 Introduction

Theory of knots [1, 2] is used in low dimensional topology and also in physics, chemistry and biology. Recently Faddeev and Niemi [3] have suggested that in certain relativistic quantum field theories knotlike configurations may appear as stable solitons. The remarkable progress in the classification of knots originated from Jones' and Vassiliev's invariants was related with the application of von Neumann algebras, Yang-Baxter equations and singularity theory, for a review see [4, 5]. Witten [6] used methods of quantum field theory in the theory of knots by considering the Wilson loops in the Chern-Simons gauge theory as knots.

In this note we describe another application of quantum field theory to the theory of knots. We consider a matrix model with d matrices $N \times N$ and show that in the limit $N \to \infty$ and $d \to 0$ the model describes the knot diagrams.

In the traditional approach to knot theory one deals with a single knot. According to Arnold's [7] and Vassiliev's [8] point of view one has to consider not a single knot but a space of all knots (cohomology of complements to discriminants). In this note from the very beginning we don't have neither knots nor a three dimensional space. We start from a matrix model in zero-dimensional space and knots appear in the limit of large matrices. The information about knots is encoded into the form of the Lagrangian of the matrix model. In some sense this approach reminds the matrix approach to superstring theory [9, 10, 11, 12] where space-time is represented as the moduli space of vacuum and strings appear in the large N limit.

2 Matrix Model and Knots

Let be given positive integers N and d and let $A_\mu = (A_\mu^{ij})$ and $B_\mu = (B_\mu^{ij})$, $i, j = 1, ..., N$ are $N \times N$ Hermitian matrices, $A_\mu^* = A_\mu$, $B_\mu^* = B_\mu$. Here $\mu = 1, ..., d$. The matrix model has the following partition function

$$Z = Z(N, d, g) = \int e^{iS} dAdB \qquad (2.1)$$

where the Lagrangian is

$$S = Tr(A_\mu B_\mu) + \frac{g}{2N} Tr(A_\mu B_\nu A_\mu B_\nu) \qquad (2.2)$$

and we assume the summation over the repeating indices. Here g is a real parameter (coupling constant) and the measure

$$dA = \prod_{\mu=1}^{d} ((\prod_{1 \le i \le j \le N} dReA_\mu^{ij}) \prod_{1 \le i < j \le N} dImA_\mu^{ij}) \qquad (2.3)$$

There exists a natural extension of the function $Z(N, d, g)$ from integers to real values of the parameter d. Actually one has the following formal expansion

$$\ln Z(N, d, g) = N^2 \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} F_{kp}(g) N^{-2p} d^k \qquad (2.4)$$

Note that $Z(N, d, 0)$ is real and therefore $\ln Z(N, d, g)$ is uniquely defined as the formal series over g. We will prove the following theorem.

Theorem. The set of connected vacuum Feynman diagrams for the model (2.1), (2.2) in the limit $N \to \infty$ and $d \to 0$ is in one-to-one correspondence with the set of alternating knot diagrams. The generating function for the alternating knot diagrams is given by the expression

$$F(g) = \lim_{d \to 0} \lim_{N \to \infty} \frac{1}{dN^2} \ln Z(N, d, g) \qquad (2.5)$$

Remark. One has the similar proposition for all (not only for alternating) knot diagrams if one takes the following Lagrangian

$$S = Tr(A_\mu A_\mu) + Tr(B_\mu B_\mu) + Tr(A_\mu B_\mu) + \frac{g}{2N} Tr(A_\mu B_\nu A_\mu B_\nu) \qquad (2.6)$$

The large N limit is considered in QCD , matrix models and superstring theory [13, 14, 15, 16, 9, 11, 12], the limit $d \to 0$ is considered in the theory of spin glasses and in polymer physics, see [17, 18].

To prove the proposition let us first remind that a *knot* is a smooth embedding of an oriented circle in oriented 3-space R^3. A collection of k pairwise disjoint knots is called a *k-link*. Two knots are equivalent (have the same isotopy type) is they are equivalent under a homeomorphism of R^3.

A knot K can be represented by a regular projection $\tilde{K}$ onto the plane having at most a finite number of transverse double points. For the plane curve $\tilde{K}$ one has to indicate which line is up ($+$) and which line is down ($-$) in an intersection point, see Fig. 1. In this way we get a graph on the plane which has 4 legs in each vertex and also has the ($+$ $-$) prescription. This graph is called the *knot diagram*. A knot diagram is called alternating if it has alternating $+$ and $-$ along a line. Two knot diagrams are called the Reidemeister equivalent if they define equivalent knotes. Reidemeister equivalence is generated by the three moves that are illustrated in Fig 2.

The fundamental group $\pi(R^3 \backslash K)$ is called the group of the knot K. The knot group is generated by simbols corresponding to legs of the knot diagram subject to some relations.

Now let us consider the integral (2.1). We obtain the Feynman diagram technique by expanding (2.1) into the formal perturbation series over the coupling constant g
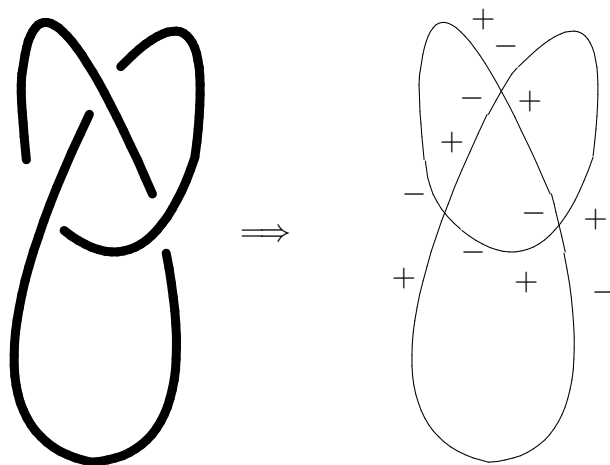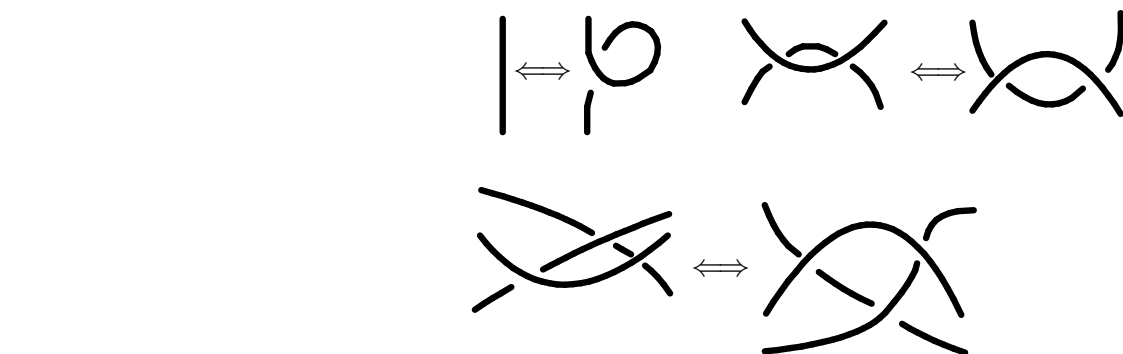
Figure 1: Trefoil
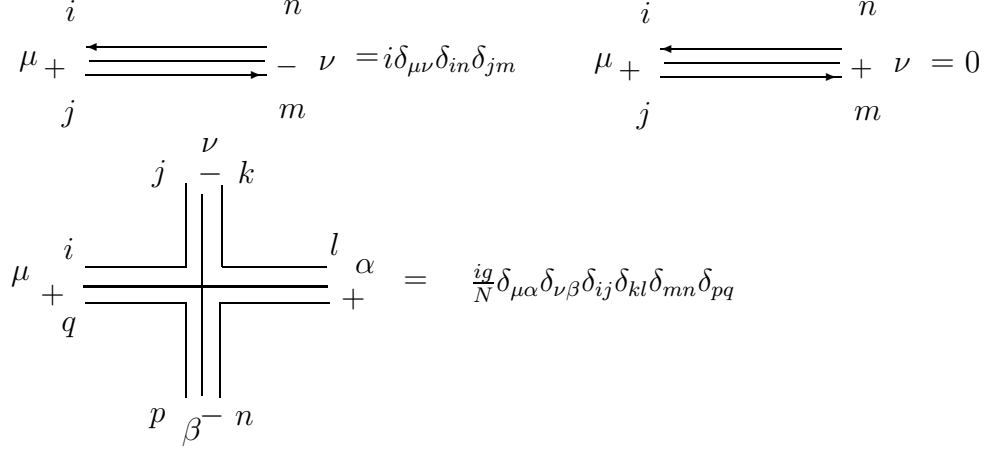

Figure 2: Reidemeister moves

$$\mu \,\, {}^{i}_{+} \, \begin{matrix} i & n \\ \rightleftarrows \\ j & m \end{matrix} \, {}^{n}_{-} \, \nu \,\, = i\delta_{\mu\nu}\delta_{in}\delta_{jm} \qquad \mu \,\, {}^{i}_{+} \, \begin{matrix} i & n \\ \rightleftarrows \\ j & m \end{matrix} \, {}^{n}_{+} \, \nu \,\, = 0$$

$$\mu \,\, {}^{i}_{+} \, {}_{q} \,\,\,\, {}^{l}_{+} \, \alpha \,\, = \,\, \frac{ig}{N}\delta_{\mu\alpha}\delta_{\nu\beta}\delta_{ij}\delta_{kl}\delta_{mn}\delta_{pq}$$

Figure 3: Elements of Feynman diagrams

and computing the corresponding Gaussian integrals

$$Z = \sum_{k=0}^{\infty} \frac{1}{k!} (\frac{ig}{N})^k \int (Tr(A_\mu B_\nu A_\mu B_\nu))^k e^{iTr(A_\mu B_\mu)} dAdB \qquad (2.7)$$

We have the following propagators and the vertex function (see Fig.3)

$$< A_\mu^{kl} B_\nu^{mn} > = i\delta_{\mu\nu}\delta^{kn}\delta^{lm} \qquad (2.8)$$

$$< A_\mu^{kl} B_\nu^{mn} > = 0, \qquad < A_\mu^{kl} B_\nu^{mn} > = 0$$

where

$$< X > = \int X e^{iTr(A_\mu B_\mu)} dAdB / \int e^{iTr(A_\mu B_\mu)} dAdB$$

The propagators are represented by triple lines each one corresponding to the separate propagation of its two indices. The middle line carries a Greek index $\mu, \nu, ...$ and all others carry Latin indices. The matrix A_μ corresponds to $+$ in Fig.3 and the matrix B_μ corresponds to $-$. To compute a contribution to the partition function Z from the $n-$th order of perturbation theory we have to draw all diagrams with n vertices, see Fig. 4 for $n = 3$.

The known connection between planarity and the large N limit [13] is based on the Euler theorem. A general Feynman diagram consists of L lines (propagators), V vertices and C closed loops of Latin indices. The contribution of the diagram is proportional to $(g/N)^V N^C = g^V N^{C-V}$. For a connected diagram one has $L = 2V$. Each closed loop of Latin index may be considered as a face of a polyhedron and the Euler relation reads $V - L + C = 2 - 2p$ where $p = 0, 1, ...$ is the number of holes of the
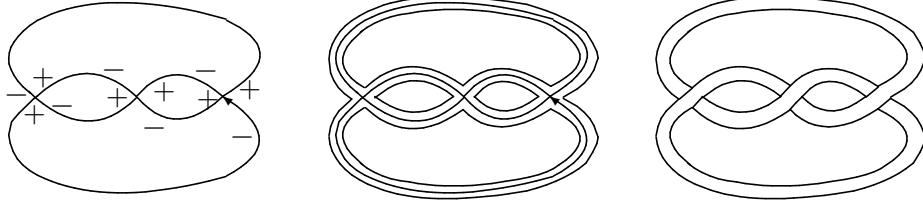
Figure 4: Third order graph (trefoil) in triple line representation

surface on which the polyhedron is drawn (genus of the Riemannian surface). Therefore $C - V = 2 - 2p$ and the contribution of the diagram is proportional to $g^V N^{2-2p}$. This justifies the factor N^{2-2p} in (2.4). We obtain that the principal contribution comes from the planar diagrams with $p = 0$. Now in the similar vine one gets that if a planar diagram has k closed loops with Greek indices (i.e. one has a k-link) then the contribution of the diagram is proportional to d^k. In particular the contribution of the knot diagrams ($k = 1$) is proportional to d. This proves the relations (2.4), (2.5).

Remark. It is interesting that the first Reidemeister move in Fig. 3 admits a natural interpretation in the Feynman diagram technique for the Lagrangian (2.2). This move generates the so called tadpole diagrams and can be removed by using the Wick normal product if we will use the Lagrangian in the normal form

$$S = Tr(A_\mu B_\mu) + \frac{g}{2N} : Tr(A_\mu B_\nu A_\mu B_\nu) : \tag{2.9}$$

where

$$: Tr(A_\mu B_\nu A_\mu B_\nu) := Tr(A_\mu B_\nu A_\mu B_\nu) - 2dNTr(A_\mu B_\mu) - 2dN^2 \tag{2.10}$$

The lowest order diagrams in matrix theory and corresponding knot diagrams are presented in Fig.6. Up to the 8-th order there is one-to-one correspondence between the matrix diagrams describing the limit $N \to \infty$, $d \to 0$ and non-isotopic knot diagrams. At the 8-th order there are 3 non-alternating knot diagrams and to reproduce them we have to consider the matrix theory with the Lagrangian (2.6).
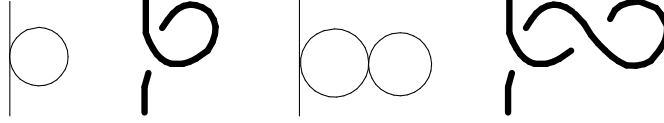
Figure 5: Tadpole diagrams

3   Knots and M(atrix) Theory

In this section we discuss the limit $d \to 0$ in M(atrix) theory. Matrix models of M-theory and superstrings are obtained by the dimensional reduction of super Yang-Mills theory in ten dimensional spacetime to p dimensions ($p = 0, 1, 2$) [9, 10, 11, 12]. The bosonic part of the Lagrangian in the matrix approach to M-theory has the form

$$S = \int (\frac{1}{2}Tr(\dot{A}_\mu \dot{A}_\mu) + \frac{g}{4N}Tr([A_\mu A_\nu][A_\mu A_\nu]))dt \qquad (3.1)$$

Here $A_\mu = A_\mu(t)$ are Hermitian $N \times N$ matrices depending on time t and $\mu = 1, ..., d$. One has

$$\frac{1}{2}Tr([A_\mu A_\nu][A_\mu A_\nu]) = Tr(A_\mu A_\nu A_\mu A_\nu) - Tr(A_\mu A_\mu A_\nu A_\nu) \qquad (3.2)$$

The first term in (3.2) has the form which has been discussed in the previous section. In the limit $d \to 0$ the principal contribution comes from the knotlike diagrams which have one loop with Greek indices. The same reasoning one can apply to the IKKT matrix model [11] with the Lagrangian

$$S = \frac{N}{2g}Tr[A_\mu, A_\nu]^2 \qquad (3.3)$$

If one makes the assumption on the existence of non-zero condencate $< A_\nu A_\nu > \sim 1$ then one gets

$$S_{eff} = \frac{N}{2g}(Tr(A_\mu A_\mu) + Tr(A_\mu A_\nu A_\mu A_\nu) - Tr(A_\mu A_\mu A_\nu A_\nu)) \qquad (3.4)$$

and one can use the described diagram technique.

It is tempting to speculate that the discussion of this note indicates that perhaps a prototypical M(atrix) without matrix theory ($d = 0$) exists in void. The eleven dimensional M-theory could be obtained from this prototypical M-theory by the de-compactification of a point.
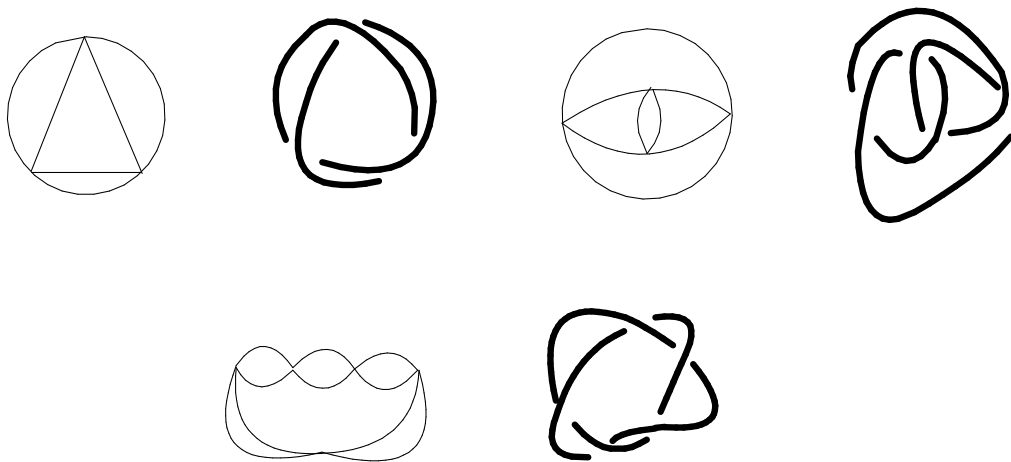
Figure 6: Feynman and knot diagrams

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